

POSITIVE QUATERNIONIC KÄHLER MANIFOLDS AND SYMMETRY RANK: II

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ABSTRACT. Let M be a positive quaternionic Kähler manifold of dimension $4m$. If the isometry group $\text{Isom}(M)$ has rank at least $\frac{m}{2} + 3$, then M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$. The lower bound for the rank is optimal if m is even.

1. INTRODUCTION

A quaternionic Kähler manifold M is an oriented Riemannian $4n$ -manifold, $n \geq 2$, whose holonomy group is contained in $Sp(n)Sp(1) \subset SO(4n)$. If $n = 1$ we add the condition that M is Einstein and self dual. Equivalently, there exists a 3-dimensional subbundle S , of the endomorphism bundle $\text{End}(TM, TM)$ locally generated by three anti-commuting almost complex structures $I, J, K = IJ$ so that the Levi-Civita connection preserves S . It is well-known [3] that a quaternionic Kähler manifold M is always Einstein, and is necessarily locally hyperKähler if its Ricci tensor vanishes. A quaternionic Kähler manifold M is called *positive* if it has positive scalar curvature. By [12] (for $n = 1$) and [19] (for $n \geq 2$, compare [15] [16]) a positive quaternionic Kähler manifold M has a twistor space a complex Fano manifold. Hitchin [12] proved a positive quaternionic Kähler 4-manifold M must be isometric to $\mathbb{C}P^2$ or S^4 . Hitchin's work was extended by Poon-Salamon [18] to dimension 8, which proves that a positive quaternionic Kähler 8-manifold M must be isometric to $\mathbb{H}P^2$, $Gr_2(\mathbb{C}^4)$ or $G_2/SO(4)$.

This leads to the Salamon-Lebrun conjecture:

Every positive quaternionic Kähler manifold is a quaternionic symmetric space.

Very recently, the conjecture was further verified for $n = 3$ in [11], using the approach initiated in [19] [18] (compare [16]). For a positive quaternionic Kähler manifold M , Salamon [19] proved that the dimension of its isometry group is equal to the index of certain twisted Dirac operator, by the Atiyah-Singer index theorem, which is a characteristic number of M coupled with the Kraines 4-form Ω (in analog with the Kähler form), and it was applied to prove that the isometry group of M is large in lower dimensions (up to dimension 16).

By [16] a positive quaternionic Kähler $4n$ -manifold M is simply connected and the second homotopy group $\pi_2(M)$ is a finite group or \mathbb{Z} , and M is isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$ according to $\pi_2(M) = 0$ or \mathbb{Z} .

An interesting question is to study positive quaternionic Kähler manifold in terms of its isometry group. This approach dates back to the work [18] for $n = 2$ [11] for $n = 3$ to proving the action is transitive, and [5] [17] for cohomogeneity one actions (and hence the isometry group must be very large). [4] classified positive quaternionic Kähler $4n$ -manifolds with isometry rank $n + 1$, using an approach on hyper-Kähler quantizations. [6] establishes a connectedness theorem and using this tool the author proved that a

positive quaternionic Kähler $4n$ -manifolds of symmetry rank $\geq n - 2$ must be either isometric to $\mathbb{H}P^n$ or $Gr_2(\mathbb{C}^{n+2})$, if $n \geq 10$.

In this paper we will combine Morse theory of the momentum map on quaternionic Kähler manifold developed in [2] and the connectedness theorem in [6] to prove the following

Theorem 1.1. *Let M be a positive quaternionic Kähler manifold of dimension $4m$. Then the isometry group $Isom(M)$ has rank (denoted by $rank(M)$) at most $(m + 1)$, and M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$ if $rank(M) \geq \frac{m}{2} + 3$.*

Since the fixed point set of an isometric circle action is either a quaternionic Kähler submanifold or a Kähler manifold. In the latter case the fixed point set has dimension at most $2m$ (the middle dimension of the manifold). Moreover, if a fixed point component is contained in $\mu^{-1}(0)$ then it must be a quaternionic Kähler submanifold, and if it is in the complement $M - \mu^{-1}(0)$ then it is Kähler (see [2]).

Theorem 1.2. *Let M be a positive quaternionic Kähler $4m$ -manifold M with an isometric S^1 -action. If N is a fixed point component of codimension 4. If $m \geq 3$ then M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$.*

The idea of proving Theorem 1.2 is as follows: by the above we know that the fixed point component $N \subset \mu^{-1}(0)$ if $4m - 4 \geq 2m + 1$ (cf. [2] Remark 3.2). Furthermore, by [8] one knows that the quantization $\mu^{-1}(0)/S^1$ has dimension at most $4m - 4$ (cf. [5]). We will prove in section 3 that $N = \mu^{-1}(0)$. Then we may use the equivariant Morse equality to show that $M = \mathbb{H}P^m$ if $b_2(M) = 0$ (cf. Lemma 4.1). By [16] this implies easily Theorem 1.2.

With the help of Theorem 1.2, the proof of Theorem 1.1 follows by using induction on the dimension and Theorems 2.1 and 2.4.

Theorem 1.1 is optimal if m is even since the rank of $\widetilde{Gr}_4(\mathbb{R}^{m+4})$ is $\frac{m}{2} + 2$. We conjecture that when m is odd, the lower bound for the rank in Theorem 1.1 may be improved by 1, that is

Conjecture 1.3. *Let M be a positive quaternionic Kähler manifold of dimension $8m + 4$. Then M is isometric to $\mathbb{H}P^{2m+1}$ or $Gr_2(\mathbb{C}^{2m+3})$ if $rank(M) \geq m + 3$.*

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2. PRELIMINARIES

In this section we recall some results on quaternionic Kähler manifolds needed in later sections.

Let (M, g) be a quaternionic Kähler manifold of dimension $4n$. Let $F \rightarrow M$ be the principal $Sp(n)Sp(1)$ -bundle over M . Locally, $F \rightarrow M$ can be lifted to a principal $Sp(n) \times Sp(1)$ -bundle, i.e. the fiberwise double cover of F . Let E, H be the locally

adjoint representations of $Sp(n)$ and $Sp(1)$ give two bundles S^2E and S^2H over M , respectively. Given the inclusion of the holonomy algebra $sp(n) \oplus sp(1)$ into $so(4n)$, the bundle $S^2E \oplus S^2H$ can be regarded as a subbundle of the bundle of 2-forms $\Lambda^2 T^*M_{\mathbb{C}}$. The bundle S^2H has fiber the Lie algebra $sp(1)$ and the local bases $\{I, J, K\}$ coresponding to i, j and $k \in sp(1)$ and consisting of three almost complex structures such that $IJ = -JI = K$.

The Kraines 4-form, Ω , associated to a quaternionic Kähler manifold M , is a non-degenerate closed form which is defined by

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

where ω_1, ω_2 and ω_3 are the locally defined 2-forms associated to the almost complex structures I, J and K . The form Ω is globally defined and non-degenerate, namely Ω^n is a constant non-zero multiple of the volume form. It is well-known that Ω is parallel if and only if M has holonomy in $Sp(n)Sp(1)$, if $n \geq 2$. Moreover, by [21] M has holonomy in $Sp(n)Sp(1)$ if and only if Ω is closed, provided $n \geq 3$.

For a quaternionic Kähler manifold, the almost complex structures may not be globally defined, e.g., the quaternionic projective spaces $\mathbb{H}P^n$ does not admit an almost complex structure. If I, J, K are integrable and covariantly constant with respect the metric, the holonomy group reduces to $Sp(n)$ and so the quaternionic Kähler manifold is hyperkähler. Wolf [22] classified quaternionic symmetric spaces of compact type, they are listed as $\mathbb{H}P^n$, the complex Grassmannian $Gr_2(\mathbb{C}^{n+2})$, and the oriented real Grassmannian $\widetilde{Gr}_4(\mathbb{R}^{n+4})$, and exactly one quaternionic symmetric space for each compact simple Lie algebra, namely $G_2/SO(4)$, $F_4/Sp(3)Sp(1)$, $E_6/SU(6)Sp(1)$, $E_7/Spin(12)Sp(1)$, $E_8/E_7Sp(1)$.

As we mentioned in the introduction, so far quaternionic symmetric spaces are the only known examples of positive quaternionic Kähler manifold.

Theorem 2.1 ([16]). *(i) (Finiteness) For any n , there are, modulo isometries and rescalings, only finitely many positive quaternionic Kähler $4n$ -manifolds.*

(ii) (Strong rigidity) Let (M, g) be a positive quaternionic Kähler $4n$ -manifold. Then M is simply connected and

$$\pi_2(M) = \begin{cases} 0, & (M, g) = \mathbb{H}P^n \\ \mathbb{Z}, & (M, g) = Gr_2(\mathbb{C}^{n+2}) \\ \text{finite with 2-torsion,} & \text{otherwise} \end{cases}$$

A submanifold N in a quaternionic Kähler manifold is called a *quaternionic submanifold* if the quaternionic structure (i.e. locally defined I, J, K) preserves the tangent bundle of N .

Proposition 2.2 ([9]). *Any quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic and quaternionic Kählerian.*

Theorem 2.3 ([6]). *Let M be a positive quaternionic Kähler manifold of dimension $4m$. Assume $f = (f_1, f_2) : N \rightarrow M \times M$, where $N = N_1 \times N_2$ and $f_i : N_i \rightarrow M$ are quaternionic immersions of compact quaternionic Kähler manifolds of dimensions $4n_i$, $i = 1, 2$. Let Δ be the diagonal of $M \times M$. Set $n = n_1 + n_2$. Then:*

(2.3.1) If $n \geq m$, then $f^{-1}(\Delta)$ is nonempty.

(2.3.2) If $n > m + 1$, then $f^{-1}(\Delta)$ is connected.

As a direct corollary of (2.3.3) we have

Theorem 2.4 ([6]). *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If $N \subset M$ is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \rightarrow M$ is $(2n - m + 1)$ -connected.*

3. HYPERKÄHLER QUOTIENT AND QUATERNIONIC KÄHLER QUOTIENT

a. Hyperkähler quotient

Let M be a hyperkähler manifold having a metric g and covariantly constant complex structures I, J, K which behave algebraically like quaternions:

$$I^2 = J^2 = K^2 = -1; IJ = -JI = K$$

Let G be a compact Lie group of isometries acting on M and preserving the structures I, J, K . The group G preserves the three Kähler forms $\omega_1, \omega_2, \omega_3$ corresponding to the three complex structures, so we may define three moment maps μ_1, μ_2, μ_3 . These can be written as a single map

$$\mu : M \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

Let

$$\mu_+ = \mu_2 + i\mu_3 : M \rightarrow \mathfrak{g}^* \otimes \mathbb{C}$$

where \mathfrak{g}^* is the dual space of the Lie algebra of G .

By [13] μ_+ is holomorphic, and so $N = \mu_+^{-1}(0)$ is a complex submanifold of M , with respect to the complex structure I , therefore N has an induced Kähler metric. By definition, $\mu^{-1}(0) = N \cap \mu_1^{-1}(0)$. The hyperkähler quotient is the quotient space $\mu^{-1}(0)/G$, denoted by $M//G$. In particular, if $\mu^{-1}(0)$ is a manifold and the induced G -action is free, then the hyperkähler quotient $M//G$ is also a hyperKähler manifold. More generally, Dancer-Swann [5] proved that the hyperkähler quotient $M//G$ may be decomposed into the union of hyperkähler manifolds, according to the isotropy decomposition of the G -action on M . However, it is not clear at all if the decomposition of $M//G$ is a stratified topological space satisfying the Goresky-MacPherson axioms, like in the symplectic quotient case [20].

In this section we will address to the structure of this decomposition. For the sake of simplicity we consider only the case of $G = S^1$ and the action is semi-free, i.e., free outside the fixed point set.

Let us start with the standard example of isometric S^1 -action on quaternionic linear space \mathbb{H}^n defined by

$$\varphi_t(u) = e^{2\pi i t} u; \quad t \in [0, 1)$$

where i is one of the quaternionic units. With global quaternionic coordinates $\{u^\alpha\}$, $\alpha = 1, \dots, n$, the standard flat metric on \mathbb{H}^n may be written as:

$$ds^2 = \sum_{\alpha} d\bar{u}^\alpha \otimes du^\alpha$$

where \bar{u}^α is the quaternionic conjugate of u^α .

The Killing vector field X of the above action is \mathbb{H} -valued:

Consider the \mathbb{H} -valued 2-form

$$\omega = \sum_{\alpha} d\bar{u}^{\alpha} \wedge du^{\alpha}$$

Observe that ω is purely imaginary since $\omega + \bar{\omega} = 0$. Note that $\omega = \omega_1 i + \omega_2 j + \omega_3 k$, where ω_i is as above.

It is easy to see that the moment map (cf. [7])

$$\mu^X = \sum_{\alpha} \bar{u}^{\alpha} i u^{\alpha}$$

Write $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$. The zero set $M_0 := \mu^{-1}(0)$ may be identified with the real algebraic variety of dimension $(4n - 3)$:

$$\{(a, b) \in \mathbb{C}^n \oplus \mathbb{C}^n : |a|^2 = |b|^2, \langle a, \bar{b} \rangle = \sum_{\alpha} a^{\alpha} b^{\alpha} = 0\}$$

The quotient M_0/S^1 is now an open cone over a $(4n - 5)$ -dimensional manifold W :

$$W = \{(a, b) \in \mathbb{C}^n \oplus \mathbb{C}^n : |a|^2 = |b|^2 = 1, \langle a, \bar{b} \rangle = \sum_{\alpha} a^{\alpha} b^{\alpha} = 0\}/S^1$$

In particular, if $n = 1$, $M_0 = \{0\}$ is a single point.

More generally, one may verify that, for any semi-free $S^1 \subset Sp(n)$ -action on \mathbb{H}^n with $\{0\}$ the only fixed point, the hyperkähler quotient $\mathbb{H}^n//S^1$ is a topological cone over a $(4n - 5)$ -dimensional real algebraic variety.

The following theorem is an analog of the Sjamaar-Lerman theorem [20] (compare [10]) for circle action in the hyperkähler case.

Theorem 3.1. *Let M^{4m} be a hyperkähler manifold with an isometric semi-free S^1 -action preserving the hyperkähler structure. If $Y_1 \subset M^{S^1}$ is a connected component of codimension $4n$ of the fixed point set in $\mu^{-1}(0)$. Then $Y_1 \subset M//S^1$ has an open neighborhood U in $M//S^1$ which is diffeomorphic to a fiber bundle over Y_1 with typical fiber a cone over a $(4n - 5)$ -dimensional algebraic variety so that Y_1 corresponds to the zero section. In particular, Y_1 is a connected component of $M//S^1$ if $n = 1$.*

Proof. It suffices to prove that, for any given point $x \in Y_1$, there is an S^1 -invariant open neighborhood U of $x \in M^{4m}$ such that $(U \cap \mu^{-1}(0))/S^1$ is a fiber bundle over $Y_1 \cap U/S^1$ with typical fiber the hyperkähler quotient $\mathbb{H}^n//S^1$.

Recall that $\omega = \omega_2 + i\omega_3$ defines a complex symplectic structure on M . The usual proof of the equivariant Darboux theorem for real symplectic manifolds applies equally well in the complex case; thus, there exists an S^1 -equivariant open ball centered at $x \in M^{4m}$, such that the complex symplectic manifold (U, ω) is equivariantly complex symplectomorphic to $(\mathbb{H}^m, i \sum_{\alpha=1}^m dz^{\alpha} \wedge dz^{\alpha+m})$, where x corresponds to the zero in \mathbb{H}^m . Identify U with \mathbb{H}^m . By assumption, the fixed point component Y_1 has codimension n , without loss of generality, we may assume that the quaternionic linear S^1 -action on \mathbb{H}^m has a fixed point set the subspace $0 \times \mathbb{H}^{m-n}$, and the action on $\mathbb{H}^n \times 0$ is given by a semi-free $S^1 \subset Sp(n)$ -action with $\{0\}$ the only fixed point. The moment map $\mu_+ : U \rightarrow \mathbb{C}$ may be identified with the moment map of the linear $S^1 \subset Sp(m)$ -action. Therefore $\mu_+^{-1}(0)$ is a complex algebraic variety, in fact $\mu_+^{-1}(0) = \bar{\mu}_+^{-1}(0) \times \mathbb{H}^{m-n}$, where $\bar{\mu}_+ : \mathbb{H}^m \rightarrow \mathbb{C}$ is the moment map of the semifree $S^1 \subset Sp(n)$ action on \mathbb{H}^n which is

Consider the variety as a symplectic manifold (at the zero, it is reducible), by the classical Darboux theorem each irreducible component may be identified with a standard symplectic ball with a semi-free linear S^1 -action, with center the only fixed point. Therefore, $\mu^{-1}(0) = \mu_1^{-1}(0) \cap \mu_+^{-1}(0)$ may be identified with $(\bar{\mu}_1^{-1}(0) \cap \bar{\mu}_+^{-1}(0)) \times \mathbb{H}^{m-n}$, where $\bar{\mu}_1$ is the moment map of the S^1 -action on $\bar{\mu}_+^{-1}(0)$, which may be identified (locally) with the zero set of the moment map of an semi-free linear $S^1 \subset U(2n-1)$ on \mathbb{C}^{2n-1} at every irreducible component, which is a cone over a $(4n-4)$ -dimensional variety (cf. [10]) Therefore, the quotient $(U \cap \mu^{-1}(0))/S^1$ fibers over $Y_1 \cap U/S^1$ with fiber a cone over a $(4n-5)$ -dimensional variety. Clearly, if $n = 1$, then the cone reduces to a single point. This proves the desired result. \square

Remark 3.2. *It seems that the neighborhood U may be chosen so that it is a fiber bundle over Y_1 with typical fiber the hyperkähler quotient H^n/S^1 , where S^1 acts on \mathbb{H}^n by the isotropy representation at the fixed point set Y_1 .*

Remark 3.3. *The same argument in the above proof extends trivially to more general situation. We will come back to this point in some future paper.*

b. Quaternionic Kähler quotient

Let M be a quaternionic Kähler manifold with non-zero scalar curvature. If G acts on M by isometries, there is a well-defined moment map, which is a section $\mu \in \Gamma(S^2 H \otimes \mathfrak{g}^*)$ solving the equation

$$\langle \nabla \mu, X \rangle = \sum_{i=1}^3 \overline{I_i X} \otimes I_i$$

for each $X \in \mathfrak{g}$; where $\bar{X} = g(X, \cdot)$ denote the 1-form dual to X with respect to the Riemannian metric. Equivalently, the above equation may be written in the following form similar to the symplectic case

$$d\mu(X) = i_X \Omega$$

A nontrivial feature for quaternionic quotient is, the section μ is uniquely determined if the scalar curvature is nonzero. Moreover, only the preimage of the zero section of the moment map, $\mu^{-1}(0)$, is well-defined.

Theorem 3.4 ([8]). *Let M^{4n} be a quaternionic Kähler manifold with nonzero scalar curvature acted on isometrically by S^1 . If S^1 acts freely on $\mu^{-1}(0)$ then $\mu^{-1}(0)/S^1$ is a quaternionic Kähler manifold of dimension $4(n-1)$.*

Since the proof of the Galicki-Lawson's theorem is local, so if the circle acts freely on a piece of the manifold is free, the same result applies well to the moment map on this piece.

Recall that a Morse function f is called *equivariantly perfect* over \mathbb{Q} if the equivariant Morse equalities hold, that is if

$$\hat{P}_t(M) = \hat{P}_t(\mu^{-1}(0)) + \sum t^{\lambda_F} \hat{P}_t(F)$$

where the sum ranges over the set of connected components of the fixed point set, λ_F is the index of F , and \hat{P}_t is the equivariant Poincaré polynomial for the equivariant cohomology with coefficients in \mathbb{Q} .

Let $f = \|\mu\|^2$. By [2] the critical set of f is the union of the zero set $f^{-1}(0) = \mu^{-1}(0)$ and the fixed point set of the circle action. Moreover, the zero set $\mu^{-1}(0)$ is connected, and a fixed point component is either contained in $\mu^{-1}(0)$ or does not intersect with $\mu^{-1}(0)$. The following result is important for this paper.

Proposition 3.6 ([2]). *Let M^{4n} be a positive quaternionic Kähler manifold acted on isometrically by S^1 . Then every connected component of the fixed point set, not contained in $\mu^{-1}(0)$, is a Kähler submanifold of $M - \mu^{-1}(0)$ of real dimension less than or equal to $2n$ whose Morse index is at least $2n$, with respect to the function f .*

For each quaternionic Kähler manifold M with non-zero scalar curvature, following [21], let $\mathfrak{u}(M)$ denote the $H^*/\{\pm 1\}$ -bundle over M :

$$\mathfrak{u}(M) = F \times_{Sp(n)Sp(1)} (H^*/\{\pm 1\})$$

where F is the principal $Sp(n)Sp(1)$ -bundle over M . Let $\pi : \mathfrak{u}(M) \rightarrow M$ denote the bundle projection. Obviously, if G acts on M by isometries, G can be lifted to a G -action on $\mathfrak{u}(M)$. It is proved in [21] that, if the scalar curvature is positive, $\mathfrak{u}(M)$ has a hyperkähler structure which is preserved by the lifted G -action. Moreover, the moment map μ is just the projection of the moment map $\hat{\mu}$ of the lifted action on $\mathfrak{u}(M)$ by π (cf. [21] Lemma 4.4).

Lemma 3.7. *Let M be a positive quaternionic Kähler manifold of dimension $4n$. Assume that S^1 acts on M effectively by isometries. Let $\mu \in \Gamma(S^2H)$ be its moment map. If $N \subset \mu^{-1}(0)$ is a fixed point component of codimension 4, then $N = \mu^{-1}(0)$.*

Proof. Let $\mathfrak{u}(M)$ be as above. By Proposition 4.2 of [5], at the fixed point $x \in N$, the isotropy representation of S^1 in $SO(3) \cong \text{Aut}(\mathfrak{u}(M)_x)$ is a finite group, where $\text{Aut}(\mathfrak{u}(M)_x)$ is the isomorphism group of the fiber at x preserving the quaternionic structure. Therefore, the preimage $\pi^{-1}(N)$ is also a fixed point component of the lifted S^1 -action on $\mathfrak{u}(M)$, which has codimension 4.

By [21] Lemma 4.4 we see that $\pi^{-1}(N) \subset \hat{\mu}^{-1}(0)$, where $\hat{\mu}$ is the moment map for the lifted S^1 -action on $\mathfrak{u}(M)$. Now S^1 acts on the normal slice of $\pi^{-1}(N)$ in $\mathfrak{u}(M)$ through a representation in $Sp(1)$. For dimension reasoning, this representation is faithful, otherwise, a finite order subgroup of S^1 acts trivially on the whole manifold $\mathfrak{u}(M)$ and so on M , a contradiction to the effectiveness of the action from our assumption. Therefore, S^1 acts semi-freely on a neighborhood of $\pi^{-1}(N)$ in $\mathfrak{u}(M)$. By now we may apply Theorem 3.1 to show that $\pi^{-1}(N)$ is a connected component of $\hat{\mu}^{-1}(0)$. Since the moment map $\hat{\mu}$ projects to the moment map μ , therefore N is also a connected component of $\mu^{-1}(0)$. By [2] $\mu^{-1}(0)$ is connected, thus $N = \mu^{-1}(0)$, the desired result follows. \square

4. PROOF OF THEOREM 1.2

Theorem 1.2 follows readily from the following Lemma and Theorem 2.1, where the dimension bound $m \geq 3$ implies that the fixed point component of codimension 4 has to be contained in $\mu^{-1}(0)$, by Proposition 3.6.

Lemma 4.1. *Let M be a positive quaternionic Kähler $4n$ -manifold with an isometric S^1 -action where $n \geq 3$. Let μ be the moment map. Assuming $b_2(M) = 0$. If $N \subset \mu^{-1}(0)$*

Proof. By Lemma 3.7 $\mu^{-1}(0) = N$, therefore S^1 acts trivially on $\mu^{-1}(0)$.

By Theorem 3.5

$$\hat{P}_t(M) = \hat{P}_t(N) + \sum_F t^{\lambda_F} \hat{P}_t(F)$$

where F runs over fixed point components outside N , and λ_F the Morse index of F . By Proposition 3.6 the Morse index $\lambda_F \geq 2n$ and are all even numbers. Thus the inclusion $N \rightarrow M$ is a $(2n - 1)$ -equivalence.

By [2] Lemma 2.2 $\hat{P}_t(M) = P_t(M)P_t(BS^1)$. Since S^1 acts trivially on F and N , we get that $\hat{P}_t(F) = P_t(F)P_t(BS^1)$ and $\hat{P}_t(N) = P_t(N)P_t(BS^1)$. The above identity reduces to

$$P_t(M) - P_t(N) = \sum_F t^{\lambda_F} P_t(F)$$

Observe that the last two terms of the left hand side is $b_2(M)t^{4n-2} + t^{4n}$.

If F is a fixed point component outside $\mu^{-1}(0)$ such that $\dim_{\mathbb{R}} F > 0$, we claim that $\dim_{\mathbb{R}} F + \lambda_F \leq 4n - 4$. Otherwise, by the above identity $\dim_{\mathbb{R}} F + \lambda_F = 4n - 2$ is impossible, and if the even integer $\dim_{\mathbb{R}} F + \lambda_F = 4n$, we conclude that the coefficients of t^{4n-2} of the right hand side is also non-zero, since F must be a compact Kähler manifold (by Proposition 3.6) and so $P_t(F)$ has nonzero coefficient at every even degree not larger than the dimension.

Clearly the identity also shows that no isolated fixed point outside $\mu^{-1}(0)$ with Morse index $4n - 2$, and there must exist an isolated fixed point with Morse index $4n$.

Put these together, by Morse theory it follows that, up to homotopy equivalence,

$$M \simeq N \cup_i e^{\lambda_i} \cup e^{4n}$$

where $2n \leq \lambda_i \leq \dim_{\mathbb{R}} F + \lambda_F \leq 4n - 4$, and e^i denotes cell of dimension i . Therefore $H^{4n-2}(M, N) = 0$. By duality $H_2(M - N) \cong H^{4n-2}(M, N) = 0$. Since the codimension of N is 4, it follows that $H_2(M) \cong H_2(M - N) = 0$. Therefore by Theorem 2.1 $M = \mathbb{H}P^n$. The desired result follows. \square

5. PROOF OF THEOREM 1.1

Let M be a positive quaternionic Kähler manifold of dimension $4m$. We call the rank of the isometry group $\text{Isom}(M)$ the symmetry rank of M , denoted by $\text{rank}(M)$. By [6] we know that $\text{rank}(M) \leq m + 1$.

Proof of Theorem 1.1. Let $r = \text{rank}(M)$. Consider the isometric T^r -action on M . Note that the T^r -action on M must have non-empty fixed point set since the Euler characteristic $\chi(M) > 0$ by [19]. Consider the isotropy representation of T^r at a fixed point $x \in M$, which must be a representation through the local linear holonomy $Sp(m)Sp(1)$ representation at $T_x M \cong \mathbb{H}^m$. If there is a stratum (a fixed point set of an isotropy group of rank ≥ 1) of codimension 4, then it must be contained in $\mu^{-1}(0)$ if $m \geq 3$ (by [2] or Proposition 3.6). By Theorem 2.1 and Lemma 4.1 the desired result follows. Thus we can assume that at x , the isotropy representation does not have any codimension 4 linear subspace fixed by some rank 1 subgroup of T^r . Let N be a maximal dimensional

By the above assumption $4m - 8 \geq \dim N \geq 2m + 4$ since $\text{rank}(N) = r - 1 \geq \frac{m}{2} + 2$, by Lemma 2.1 of [6]. Note that N is a quaternionic Kähler manifold since $N \subset \mu^{-1}(0)$. By Theorem 2.4 we see that $\pi_2(N) \cong \pi_2(M)$. By Theorem 2.1 it suffices to prove $\pi_2(N) = 0$ or \mathbb{Z} . By induction we may consider T^r -action on N , and applying Lemma 4.1 once again. Finally it suffices to consider the case where a 16-dimensional quaternionic Kähler submanifold of M , M^{16} , with an effective isometric action by torus of rank ≥ 5 . In this case there is a quaternionic Kähler submanifold $M^{12} \subset M^{16}$ fixed by a circle group (cf. [6]). By Lemma 4.1, Theorem 2.1 and Theorem 2.4 $M^{16} = \mathbb{H}P^4$ or $\text{Gr}_2(\mathbb{C}^6)$, the desired result follows.

Case (ii): If $m = 1(\text{mod } 2)$.

Similar to the above $\dim N \geq 2m + 6$ for the same reasoning. By Theorem 2.4 $\pi_2(N) \cong \pi_2(M)$. The same argument by induction reduces the problem to the case of a quaternionic Kähler submanifold of dimension 20, M^{20} , with an effective isometric torus action of rank ≥ 6 . Once again the argument in [6] shows that M^{20} has a quaternionic submanifold M^{16} of rank ≥ 5 . By (i) we see that $M^{16} = \mathbb{H}P^4$ or $\text{Gr}_2(\mathbb{C}^6)$. By Theorem 2.1 and Theorem 2.4 again we complete the proof. \square

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